§ Varieties

$$X \overset{o}{\hookrightarrow} V \overset{c}{\hookrightarrow} \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \times \mathbb{A}^m$$

V irr.

Variety := open subset in a nonempty irreducible algebraic set in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \times \mathbb{A}^m$

Induced topology on subset $Y \subseteq X$: $\mathcal{T}_Y := \{ U \cap Y \mid U \in \mathcal{T}_X \}$

i.e. $Z \overset{c}{\hookrightarrow} Y$ ($Z \overset{o}{\hookrightarrow} Y$) $\Leftrightarrow \exists\, Z' \overset{c}{\hookrightarrow} X$ ($Z' \overset{o}{\hookrightarrow} X$) s.t. $Z = Z' \cap Y$

closure dense continuous homeomorphism

Zariski topology on variety := induced top.

Fact: X = variety. Then

1) X = irr.

2) $\forall$ open $U \neq \phi$ is dense.

3) $\forall$ open $U_1, U_2 \neq \phi \Rightarrow U_1 \cap U_2 \neq \phi$.

Pf: 1) $X = X_1 \cup X_2$ $\quad X_i \overset{c}{\hookrightarrow} X$ $\left(\Rightarrow \exists\, Z_i \overset{c}{\hookrightarrow} V \text{ s.t. } X_i = Z_i \cap X \right)$

$V = Z_1 \cup Z_2 \cup (V \setminus X) = \text{irr} \Rightarrow V = Z_1$ or $V = Z_2$ or $Z \cancel{=} V \setminus X$

$\Rightarrow X = X_1$ or $X = X_2$

2) $X = \overline{U} \cup (X \setminus U) \Rightarrow \checkmark$

3) if not. $\Rightarrow X = (X \setminus U_1) \cup (X \setminus U_2)$ ↯

$U \overset{\circ}{\hookrightarrow} X \Rightarrow U$ = var. open subvariety.

Fact (closed subvariety) Y = irreducible closed subset of X.

Then Y is also a variety.

Pf: $Y \hookrightarrow X \Rightarrow \exists Z$ s.t. $\rightrightarrows$

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, hook] & Z \arrow[d, hook] \\ X \arrow[r, hook] \arrow[d, hook] & V \arrow[d, hook] \\ U \arrow[r, hook] & \mathbb{P} = \mathbb{P}^{n_1} \times \cdots \mathbb{P}^{n_r} \times \mathbb{A}^m \end{tikzcd}$$

may not irr. !

$\overline{Y}$:= closure of Y in $\mathbb{P}$

$Y \subseteq \overline{Y} \hookrightarrow \mathbb{P}$ We only need ~~to~~ show $Y \overset{\circ}{\hookrightarrow} \overline{Y}$

$$Y \subseteq U \cap \overline{Y} \subseteq U \cap Z = (U \cap V) \cap Z = X \cap Z = Y.$$

$\Rightarrow Y \overset{\circ}{\hookrightarrow} \overline{Y}$.

$X \overset{\text{op}}{\hookrightarrow} V \hookrightarrow \mathbb{P} = \mathbb{P}^{n_1} \times \cdots \mathbb{P}^{n_r} \times \mathbb{A}^m$

$$k(V) := \left\{ \frac{F \bmod I(V)}{G \bmod I(V)} \;\middle|\; F, G \text{ multiforms of the same degree} \right\}$$

$$\begin{array}{l} \mathcal{O}_P(V) := \{ h \in k(V) \mid h \text{ is defined at } P \} \\ \quad \cup\!\text{I} \\ m_P(V) \end{array}$$

$$V = \text{affine} \Rightarrow \Gamma(V) = \bigcap_{P \in V} \mathcal{O}_P(V)$$

$$\left\{ \begin{array}{l} k(X) := k(V) \\ \mathcal{O}_P(X) := \mathcal{O}_P(V) \\ \Gamma(U) := \bigcap_{P \in U} \mathcal{O}_P(X) \quad \forall\, U \overset{\text{op}}{\subset} X. \end{array} \right.$$

Fact: 1) $\Gamma(U) \subseteq k(X)$ subring

2) $U' \subset U \Rightarrow \Gamma(U') \supset \Gamma(U)$

3) if X is affine, then $\Gamma(X) =$ Coordinate ring of X

ring homomorphism $\Gamma(U) \longrightarrow \mathcal{F}(U, k)$

$\{ k\text{-valued functions on } U \}$

$$z \longmapsto \left(\begin{array}{ccc} U & \longrightarrow & k \\ P & \longmapsto & z(P) \end{array} \right)$$

To consider $\Gamma(U)$ as a ring of functions on U, we need to show the map is injective.

Prop 1: $X =$ variety, $\emptyset \neq U \overset{\circ}{\subset} X$, $z \in \Gamma(U)$.

$$z(P) = 0 \quad \forall P \in U \Rightarrow z = 0.$$

Pf: 1° $X =$ affine i.e. $X \overset{c}{\subset} \mathbb{A}^n$

$z = \frac{f}{g}$ with $f = F \bmod I(X)$ & $g = G \bmod I(X) \neq 0$.

$U' := \{p \in U \mid g(p) \neq 0\} \neq \emptyset \quad (\overset{\circ}{\subset} U)$

$z|_U = 0 \Rightarrow z|_{U'} = 0 \Rightarrow f|_{U'} = 0 \Rightarrow F|_{U'} = 0$

$\forall H \in I(X \backslash U') \Rightarrow FH|_X = 0 \Rightarrow FH \in I(X) =$ prime

$\Rightarrow F \in I(X)$ or $\cancel{H \in I(X)}$

$\Rightarrow f = 0 \in \Gamma(X) \Rightarrow z = 0$

2° $U \overset{\circ}{\subset} X \overset{c}{\subset} \mathbb{P} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \times \mathbb{A}^m$

$P \in U \cap V \overset{\circ}{\subset} X \cap V \overset{c}{\subset} U_{i_1} \times \cdots \times U_{i_r} \times \mathbb{A}^m =: V \ni P$, $= \mathbb{A}^{n_1 + \cdots + n_r + m}$

$z|_U = 0 \Rightarrow z|_{U \cap V} = 0 \overset{1°}{\Rightarrow} z = 0 \in \Gamma(U \cap V) \Rightarrow z = 0 \in \Gamma(U)$

3° $U \overset{\circ}{\subset} X \overset{\circ}{\subset} V \overset{c}{\subset} \mathbb{P}$

2° $\Rightarrow$ ✓